

## Damage Factor Assessment in Plate Structures Using Finite Element Method

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**ABSTRACT.** Using finite element analysis, damage factor ( $DF$ ) for damaged plate structure is presented. The damage factor ( $DF$ ) is able to localize the damage severity in the plate by a clear “jump” at the location of the damage in plate. The magnitude of the damage factor ( $DF$ ) presented here varies linearly with the exact damage measure ( $DM$ ) imposed in the plate. A new intuitive mode dependent factor is introduced. The mode dependent factor  $\alpha$  is to improve the damage factor ( $DF$ ) to be able to quantify the severity of the damage in plate.

**KEYWORDS:** modal analysis, damage detection, localization, plate vibrations, Damage Factor, Damage Measure.

### 1. Introduction

Structures are prone to damage during their service lives. Damage is caused by factors such as corrosion, fatigue, impact and overloads. Structural damage causes deviations of geometric or material properties from nominal or baseline values. For the sake of safety, reliability and operational life, it is essential to monitor the health status of structural systems. To this end the availability of suitable techniques for nondestructive damage detection in aerospace, civil and mechanical engineering structures are essential. Experimental modal analysis has become an increasingly accepted method for determining the overall health of a structure. It uses parameters based on differences of measured modal properties from baseline or nominal values to identify the location of damage. Until recently structural damage identification efforts were focused on damage

detection by spotting locations of reduced natural frequency. Thus Adams *et al.* <sup>[1]</sup> used the decrease in natural frequencies and increase in damping to detect cracks in fiber-reinforced plastics. They developed a theoretical model to predict the damage and its location based on receptance analysis. The analysis was done by using axial modes of vibration, and it is valid for structures which can be treated as one-dimensional.

Vandiver <sup>[2]</sup> and Loland *et al.* <sup>[3]</sup> used the same principle to detect damage in offshore structures. From relative changes in the natural frequencies of different modes, Loland *et al.* <sup>[3]</sup> reported that they could predict the location of the damage. They demonstrated the use of their technique on oil drilling platforms in the North Sea. Cawley *et al.* <sup>[4]</sup> employed sensitivity analysis to deduce the location of damage in two-dimensional structures, based on a finite element analysis method. Flexural modes of vibration were used in this case. The method was applied to the case of a flat plate with the assumption that the modulus of elasticity in the damage area becomes equal to zero. For each element of the model, the sensitivity to the change was evaluated. The result of the analysis agreed well with the experimental results. The drawback of this method was that a large amount of computation needs to be performed subsequent to data collection to predict the location of the damage.

Silva *et al.* <sup>[5]</sup> performed extensive experimental dynamic analyses of free-free beams for the prediction of location and depth of cracks in straight beams. Cracks were simulated by cuts made using a very thin cutting tool, and real cracks were obtained by means of a three-point-bending fatigue technique. Variation in frequency tended to be higher in cracked beams than in slotted ones. Ostachowicz *et al.* <sup>[6]</sup> studied the effect of two open cracks on the natural frequencies of a cantilever beam. This was done numerically with some success.

Recently, investigators discovered that variations in mode shape curvature are more sensitive to damage than the reduction in natural frequency. Thus Sanders *et al.* <sup>[7]</sup> developed a theory to detect and locate damage in structures made of fiber-reinforced composites in which they used modal sensitivity equations in conjunction with internal-state variable constitutive theory. Choi *et al.* <sup>[8]</sup> compared the results of two damage detection techniques in a computer-simulated 2-D plate experiment. The first method is the Local Compliance Method, which is derived from classical plate theory. The second method is the Modal Strain Energy method, which is derived using expression for strain energy of a thin plate. The authors reported that the performance of the Local Compliance Method had more inherent noise because of fourth order numerical differentiation of data and other aspects of its numerical methods.

Petro *et al.* <sup>[9]</sup> tested a free-free cantilever beam using modal strain energy

techniques and reported satisfactory results in damage identification. The beam was tested without damage, and then tested after damage was introduced. An 'intuitive' damage parameter called the Strain Energy Damage Index (SEDI) was coined. SEDI measured relative modal strain energy changes due to damage, based on one-dimensional static beam mechanics. This technique required numerically differentiating modal test data twice to obtain the curvature of the mode shape and then numerically integrating to find the strain energy variation along the beam. In spite of the potential error during numerical analysis and difficulties posed by noise, this effort was reported to furnish satisfactory results in locating the damage. The SEDI parameter exhibited a jump where damage was introduced. It did not, however, serve to assess the severity of the damage.

Osegueda *et al.* <sup>[10]</sup> defined a new modal strain energy parameter, the Strain Energy Damage Index 2 (SEDI2). To this end a set of four double cantilever beam specimens with damage in various locations was tested, with varying levels of success. Meza *et al.* <sup>[11]</sup> implemented Modal Strain Energy Techniques and a scanning laser vibrometer to test a DC-9 aircraft forward fuselage for different artificially induced damage scenarios. These tests also had varying degrees of success. In some modes, the correct damaged area was indicated and in other modes an incorrect area was indicated. These investigators then averaged all strain energy differences from all different modes, and claimed success in isolating the damage.

Carrasco *et al.* <sup>[12,13]</sup> used Modal Strain Energy Techniques to identify damage in a relatively complicated truss structure with several types of damage scenarios. This effort weighted the strain energy differences in elements of the finite element model in proportion to the strain energy distribution in the undamaged case. Noise effects were compensated by assigning large weights to the sensitive elements and small weights to the insensitive elements. They reported varying levels of success. Cornwell *et al.* <sup>[14,15]</sup> applied Modal Strain Energy Techniques for beam-like structures and plate like structures. Theoretical development of the Modal Strain Energy Technique for each case (beam and plate) was presented. It investigated various degrees of damage. The study had varying degrees of success depending on the different combinations of parameters that are varied.

Gawronski *et al.* <sup>[16]</sup> used modal and sensor norms to determine damage locations. Their approach investigated the localization of damaged elements of a structure. Nicholson, and Alnefaie <sup>[17]</sup> introduced a new damage sensitive parameter called the Modal Moment Index (MMI). It uses the difference in the modal strain energies of the corresponding undamaged and damaged elements, but assumes that the "modal moment" is unaffected by damage. It is

successfully applied for the beam structures. Pai *et al.* [18] studied the boundary effect detection method for pinpointing locations of small damages in beams using operational deflection shapes measured by a scanning laser vibrometer. Ray *et al.* [19] demonstrated the concept of sensitivity enhancing control to aid in damage detection in smart structures through both experimental and simulation evaluation. They implemented state estimate feedback methods using strain measurements along the structure.

Hu *et al.* [20] developed identification algorithms for assessing structural damages using modal test data. Kessler *et al.* [21] used a frequency response function method for detecting a small damage in a simple composite structure. Recently, Abdo, and Hori [22] presented a numerical study for the establishing of relationships between damage characteristics and changes in dynamic properties of a structure. They proposed that the rotation of mode shape has the characteristic of localization at the damaged region. They reported that this technique is capable of locating multiple damage locations with different sizes.

In what follows, a damage factor ( $DF$ ) to localize and quantify the damage in plate is introduced. It is shown that the damage factor ( $DF$ ) not only exhibits a sharp jump at the location of damage, but that its magnitude varies linearly with the damage measure ( $DM$ ). Eventually, the damage factor ( $DF$ ) is further improved by introducing a mode dependent factor  $\alpha$ .

## 2. Analysis

A cantilever plate is modeled using 100 rectangular elements as shown in Figure (1). We suppose that damage primarily affects the average elastic modulus in the elements, with minimal effects on the cross-section or the mass density. The average elastic moduli before and after damage are  $E$  and  $E^*$ . A pointwise damage measure ( $DM$ ) is now introduced as

$$DM = \frac{D - D^*}{D} \quad (1)$$

Where  $D$  is the element plate stiffness given by

$$D = \frac{Eh^3}{12(1 - \nu^2)} \quad (2)$$

Then simulated damage is imposed in element (46) as a relative decrease in stiffness (i.e.  $DM$  from 20% to 80% at element 46). For example if the damage measure  $DM$  is 80% at element 46 then

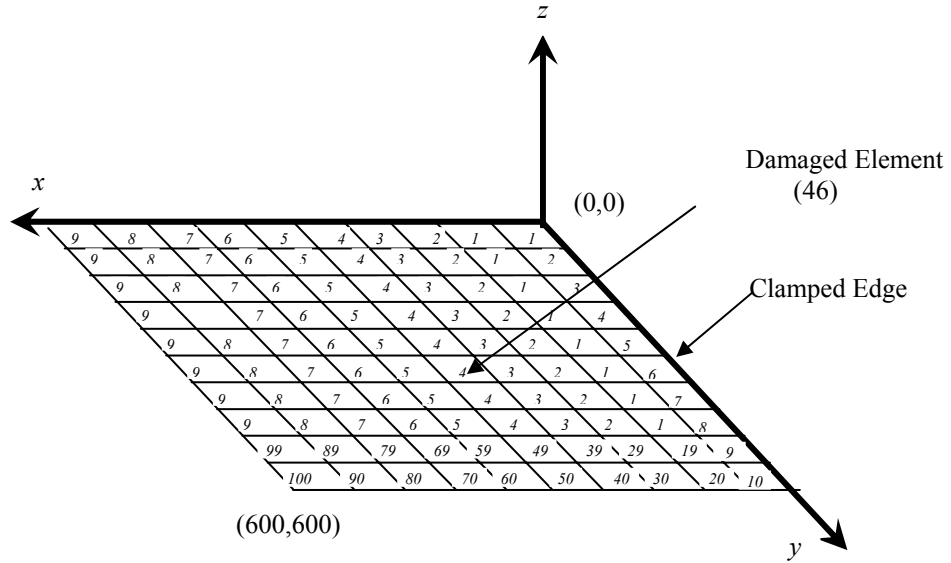


Fig. 1. Model of the Cantilever Plate of 100 Elements with one damaged element.

$$0.8 = \frac{D - D^*}{D} \tag{3}$$

It means that the stiffness of the damaged element 46 is 20% of the undamaged element which is given by

$$D^* = 0.2 * D \tag{4}$$

The material of the plate used in the finite element simulation is assumed to be Aluminum 6061 sheet. The plate dimensions are 600 mm length, 600 mm width and 3mm thickness. The plate modulus of elasticity is  $E = 70 \times 10^3$  MPa and the mass density  $\rho = 2710$  kg /m<sup>3</sup>.

### 2.1 The Modal Strain Energy of a Plate-Like Structure

The strain energy of the plate is

$$U = \frac{D}{2} \int_0^b \int_0^a \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) + 2(1 - \nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 dx dy \tag{5}$$

where  $D$  is the plate stiffness,  $h$  is the plate thickness and  $\nu$  is the Poisson's ratio.

For the  $i^{\text{th}}$  mode shape  $\Phi_i(x,y)$  the modal strain energy associated with that mode shape is

$$U = \frac{D}{2} \int_0^b \int_0^a \left( \frac{\partial^2 \Phi_i}{\partial x^2} \right)^2 + \left( \frac{\partial^2 \Phi_i}{\partial y^2} \right)^2 + 2\nu \left( \frac{\partial^2 \Phi_i}{\partial x^2} \right) \left( \frac{\partial^2 \Phi_i}{\partial y^2} \right) + 2(1-\nu) \left( \frac{\partial^2 \Phi_i}{\partial x \partial y} \right)^2 dx dy \quad (6)$$

Referring to Figure (2), if the plate is divided into  $N_x$  sub-divisions in the  $x$  direction and  $N_y$  sub-divisions in the  $y$  direction, then the modal strain energy of sub-region  $jk$  (plate element) for  $i^{\text{th}}$  mode is given by

$$U_{ijk} = \frac{D_{jk}}{2} \int_{b_k}^{b_{k+1}} \int_{a_j}^{a_{j+1}} \left( \frac{\partial^2 \Phi_i}{\partial x^2} \right)^2 + \left( \frac{\partial^2 \Phi_i}{\partial y^2} \right)^2 + 2\nu \left( \frac{\partial^2 \Phi_i}{\partial x^2} \right) \left( \frac{\partial^2 \Phi_i}{\partial y^2} \right) + 2(1-\nu) \left( \frac{\partial^2 \Phi_i}{\partial x \partial y} \right)^2 dx dy \quad (7)$$

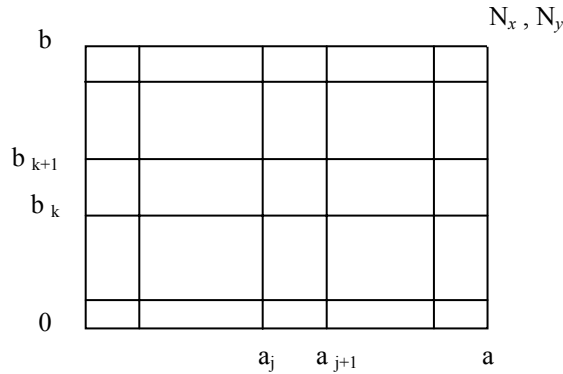


Fig. 2. A schematic of a plate's sub-regions.

In matrix form

$$U_{ei} = \frac{I}{2} \{ \delta_{ei} \}^T [K_e] \{ \delta_{ei} \} \quad (8)$$

where  $\{ \delta_{ei} \}$  is a  $(1 \times 12)$  vector which contains the nodal displacements and the rotations about the  $x$  and  $y$  axes for each node of the plate element, at the  $i^{\text{th}}$  mode.

### 2.2 The Finite Element Model of a Plate-Like Structure

Suppose that  $w(x,y)$  is interpolated by expressions of the form

$$w(x,y) = \sum_{j=1}^n \delta_j \Psi_j(x,y) \tag{9}$$

where  $\delta_j$  denote the nodal values of  $w$  and its derivatives, and  $\Psi_j(x,y)$  are Hermite interpolation functions [23]. Figure (1) shows the finite element mesh of the cantilevered plate model in which a rectangular plate element is used. The plate element has four nodes and each node has three degrees of freedom ( $w, \theta_x, \theta_y$ ) as shown in Figure (3).

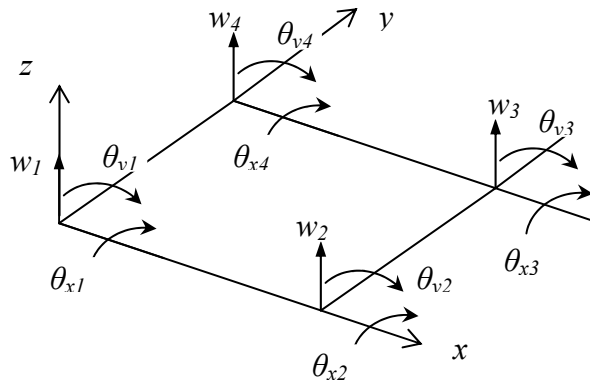


Fig. 3. Plate Element.

Now the finite element model in matrix form is

$$[M^e] \{ \delta^e \} + [K^e] \{ \delta^e \} = \{ f^e \} + \{ Q^e \} \tag{10}$$

where

$$k_{ij}^e = \int_{\Omega^e} \left[ D \frac{\partial^2 \Psi_i}{\partial x^2} \frac{\partial^2 \Psi_j}{\partial x^2} + \nu D \left( \frac{\partial^2 \Psi_i}{\partial x^2} \frac{\partial^2 \Psi_j}{\partial y^2} + \frac{\partial^2 \Psi_i}{\partial y^2} \frac{\partial^2 \Psi_j}{\partial x^2} \right) + D \frac{\partial^2 \Psi_i}{\partial y^2} \frac{\partial^2 \Psi_j}{\partial y^2} + 2(1-\nu) D \frac{\partial^2 \Psi_i}{\partial x \partial y} \frac{\partial^2 \Psi_j}{\partial x \partial y} \right] dx dy \tag{11}$$

$$M_{ij}^e = \int_{\Omega^e} \left[ I_0 \Psi_i \Psi_j + I_2 \left( \frac{\partial \Psi_i}{\partial x} \frac{\partial \Psi_j}{\partial x} + \frac{\partial \Psi_i}{\partial y} \frac{\partial \Psi_j}{\partial y} \right) \right] dx dy \tag{12}$$

$$f_i^e = \int_{\Omega^e} f \Psi_i \, dx dy \quad (13)$$

$$Q_i^e = \oint_{\Gamma^e} \left( -M_n \frac{\partial \Psi_i}{\partial n} + V_n \Psi_i \right) ds \quad (14)$$

Here  $M_n$  and  $V_n$  are the edge loads,  $I_o = \rho h$ , and  $I_2 = \rho h^3/12$ . For natural vibration, the eigenvalue problem is given by

$$\left( [K^e] - \omega_i^2 [M^e] \right) \{\delta_o\} = \{0\} \quad (15)$$

After solving the eigenvalue problem to obtain the eigenvalues and the eigenvectors the modal strain energies for each element at the  $i^{\text{th}}$  mode is computed using equation (8).

$$U_{ei} = \frac{1}{2} \{\delta_{ei}\}^T [K_e] \{\delta_{ei}\}$$

### 2.3 The Damage Factor (DF) of Plate Structure

A cantilever plate is modeled using 100 rectangular elements. The modal strain energy of each element at the  $i^{\text{th}}$  mode is calculated before damage by equation (8)

$$U_{ei} = \frac{1}{2} \{\delta_{ei}\}^T [K_e] \{\delta_{ei}\}$$

Then simulated damage is introduced in element (46) as a relative decrease in stiffness (i.e. as  $D$  from 20% to 80%). The modal strain energy of each element at the  $i^{\text{th}}$  mode is calculated after the damage is imposed by

$$U_{ei}^* = \frac{1}{2} \{\delta_{ei}^*\}^T [K_e] \{\delta_{ei}^*\} \quad (16)$$

where  $\{\delta_{ei}^*\}$  is a (1 x 12) damage vector which contains the damage nodal displacements and the damage rotations about the  $x$  and  $y$  axes for each node of the plate element, at the  $i^{\text{th}}$  mode.

Then the damage factor  $DF^{[17]}$  is computed for each element at the  $i^{\text{th}}$  mode for modes one through six as



$$DF = 1 - \sqrt{\frac{U_{ei}}{U_{ei}^*}} \quad (17)$$

Eventually, the improved damage factor (*IDF*) is computed for each element at the  $i^{th}$  mode for modes one through six as

$$IDF = \alpha \left( 1 - \sqrt{\frac{U_{ei}}{U_{ei}^*}} \right) \quad (18)$$

where  $\alpha$  is the mode-dependent factor.

OR

Improved Damage Factor = mode-dependent factor \* Damage factor

Each mode of the plate has a corresponding damage factor (*DF*). Obviously, the relationship between the damage factor (*DF*) and the damage measure (*DM*) is linear. Accordingly, the mode dependent factor  $\alpha$  is chosen in such away that the computed damage factor (*DF*) is closely related to the damage measure (*DM*). This mode dependent factor  $\alpha$  is different from mode to mode. In a future work the damage factor (*DF*) (21) can be modified to obtain a fixed mode dependent factor  $\alpha$ .

### 3. Results

The cantilever plate is modeled using 100 rectangular elements. The modal strain energies for each element are calculated before damage; then simulated damage is introduced in element (46) as a relative decrease in stiffness (i.e. damage measure (*DM*) from 20% to 80%). The damage factor (*DF*) of each element is calculated for modes one through six. Figures (4) through (9) shows a plot of the damage factor (*DF*) using damage measure (*DM*) = 20%. In these Figures (4) through (9) the  $z$  - axes represents the damage factor (*DF*) at each element, the  $x$  - axes and the  $y$  - axes represents the location of the center of each element of the plate.

Figure (10) shows the damage factor (*DF*) at element 46 for modes one through six versus the imposed damage measure (*DM*) at element 46. Figure (11) shows the improved damage factor (*IDF*) at element 46 for modes one through six versus the imposed damage measure (*DM*) at element 46. The damage factor (*DF*) is improved by an “intuitive” mode-dependent factor  $\alpha$

multiplied by the damage factor ( $DF$ ) at element 46. The linear dependence on damage is considered between the damage factor ( $DF$ ) and damage measure ( $DM$ ).

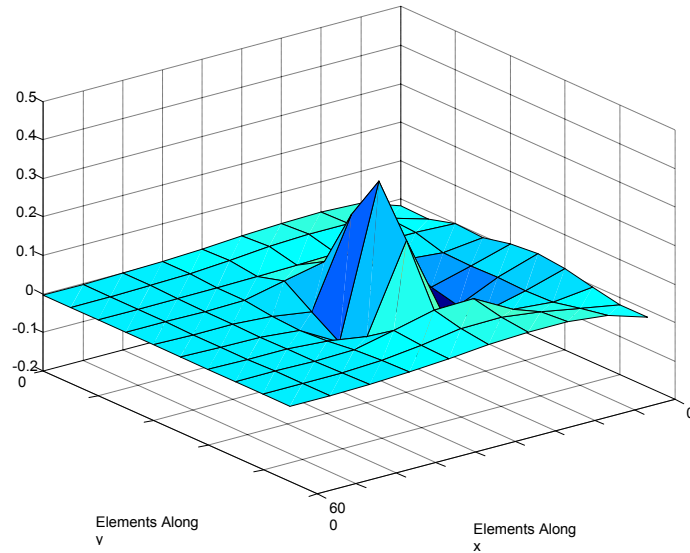


Fig. 4. Damage factor at each element of the plate for the first mode.

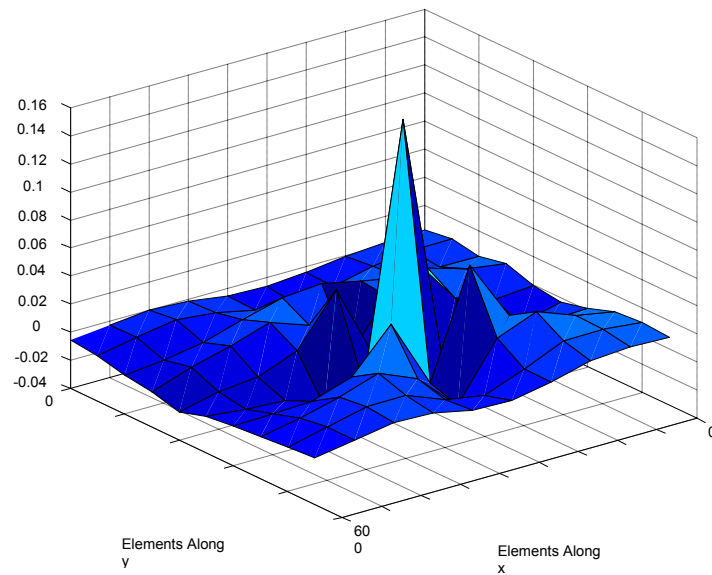


Fig. 5. Damage factor at each element of the plate for the second mode.

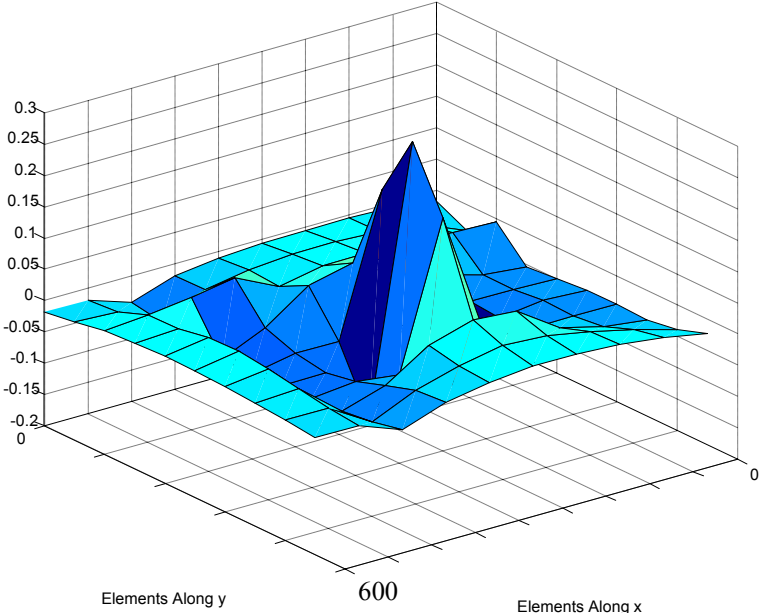
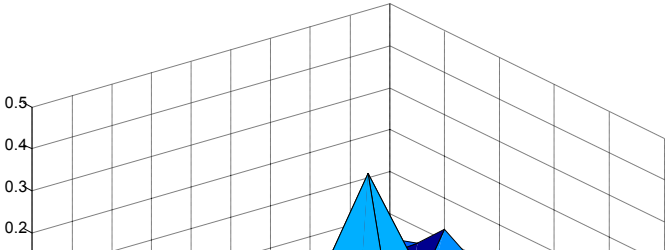


Fig. 6. Damage factor at each element of the plate for the third mode.



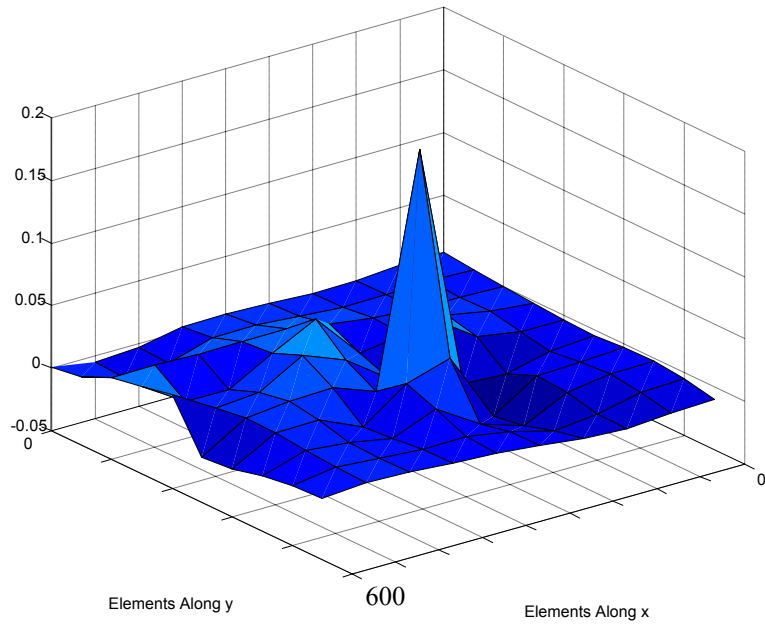


Fig. 8. Damage factor at each element of the plate for the fifth mode.

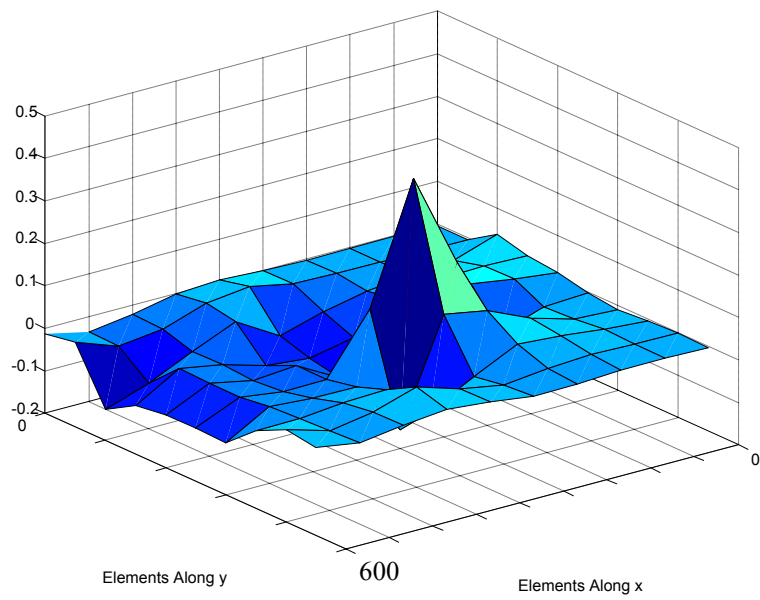


Fig. 9. Damage factor at each element of the plate for the sixth mode.

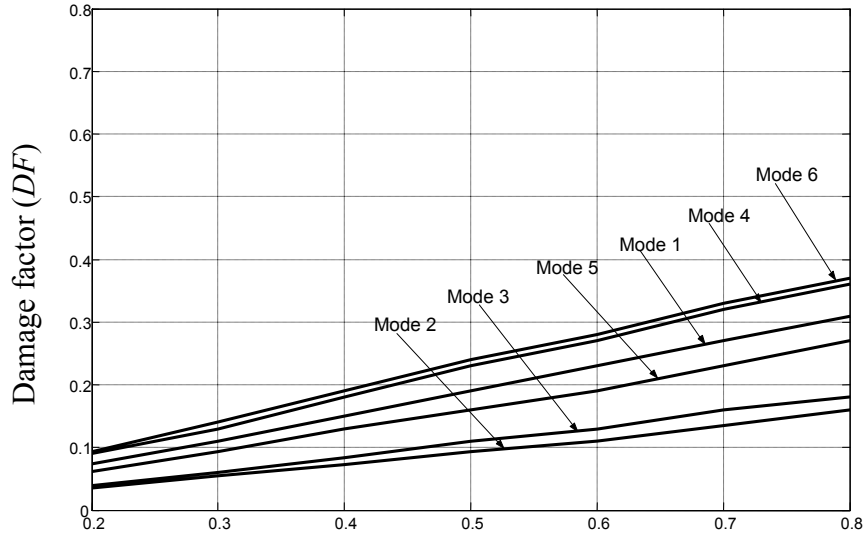


Fig. 10. Damage factor at the damaged element 46 (20% - 80%) of the plate for modes one through six.

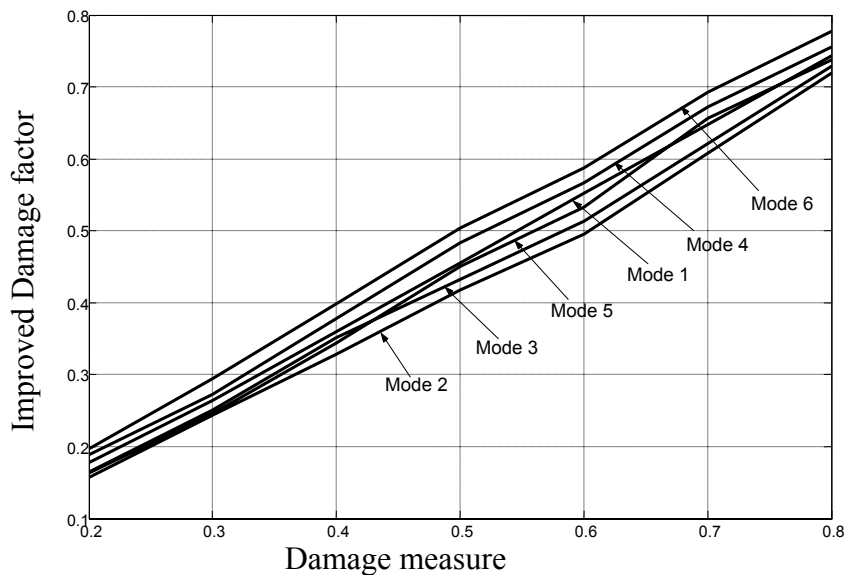


Fig. 11. Improved Damage factor at the damaged element 46 (20% - 80%) of the plate for modes one through six.

#### 4. Conclusion

Finite element simulations on damaged plate structures are presented. The damage factor ( $DF$ ) for damaged plate *proves to be closely related to the level of damage* measured by a relative modulus decrease, and it jumps sharply at the damage site. Damage factor ( $DF$ ) shows a promising damage assessment factor in plate structures. Although, damaged is localized by the damage factor ( $DF$ ) its value is related to the damage measure through a linear relationship between the damage factor ( $DF$ ) and the damage measure ( $DM$ ). Eventually, the damage factor ( $DF$ ) is improved by a mode-dependent factor  $\alpha$ . In a future work the damage factor ( $DF$ ) (21) can be modified to obtain a fixed mode dependent factor  $\alpha$ .

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#### Nomenclature:

$A$	cross-sectional area
$D$	bending stiffness
$e$	element index
$E$	elastic modulus
$f$	consistent force
$I$	bending moment of area
$[K^e]$	global stiffness matrix
$[K_e]$	Element stiffness matrix
$[M]$	consistent mass matrix
$M$	moment
$U$	strain energy
$V$	also shear force
$t$	thickness
$w$	z-displacement
$x,y$	coordinates
$\nu$	Poisson's ratio
$\rho$	mass density
$\Phi_j$ <i>j</i> th	eigenvector, mode shape
$\delta$	nodal displacements
$\Psi$	Hermite interpolation function
$\omega$	natural frequency

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